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Hernán Larralde, Pablo Jensen, Margaret Edwards

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Hernán Larralde (1), Pablo Jensen (2,3,4), Margaret Edwards (4)
(1) Instituto de Ciencias Fisicas, Universidad Nacional Autónoma de México, Apdo. Postal 48-3, C.P. 62251, Cuernavaca, Morelos, MEXICO (2) Laboratoire de Physique, UMR CNRS 5672, Ecole Normale Suprieure de Lyon, 69007 Lyon Cedex, FRANCE (3) Institut Rhne-Alpin des Systmes Complexes, GIS CNRS, Lyon, FRANCE (4) Laboratoire d'Economie des Transports, UMR CNRS 5593, Université Lyon-2, FRANCE

We present an analytical solution to a two dimensional Hotelling model with quadratic transportation costs for two stores in a square city. We assume that the consumers choice as to which store to patronize is tempered by a logit function. As in the one-dimensional case, stores are led to aggregate spatially as the disorder introduced by the logit increases. This solution is confirmed by numerical simulations.

I. INTRODUCTION

Hotelling's model [1] is one of the preferred "toy models" of spatial economics. Variations of this model and its "Principle of Minimum Differentiation" have generated a large literature (for reviews, see [2–5], for recent papers, see [6–8]) allowing researchers to play with elementary mechanisms at will. In essence, this model consists in "consumers" that are distributed in a bounded space and choose to buy at the store which maximizes their utility. The definition of the consumer utility includes store prices and transportation costs. The aim of the model is to determine optimal locations and prices for competing stores trying to maximize their profits.

The main conclusions of the one dimensional Hotelling model for two stores looking for "location then prices" optima are summarized in [2]. If the consumers choose systematically the most convenient store, i.e. that which represents the lowest total cost, then stores prefer to locate near the edges of the segment, because this allows them to exert some "market power" on the consumers located close to them. Indeed, when the stores are located near each other, consumers have similar traveling costs for both stores and a slightly lower price is sufficient for one of the stores to collect most of the demand. Therefore, competition drives both stores to minimal prices and zero profits. In contrast, when the stores are located far from each other, consumer's choice is mainly determined by the distance to each store. Therefore the stores can sell at higher prices and obtain a higher profit than in the previous case.

The situation changes when some "disorder" is introduced in the consumer's preferences. This disorder represents additional choice factors, not taken into account explicitly in the analysis, and therefore entering the picture only as a random "noise". Usually, the effect of the random factor in the consumer's choice is introduced through a logit [2] function. This randomness in consumer's preferences renders their choice less sensible to stores' characteristics and the advantage of spatial differentiation is lessened. Then stores move towards the centre, which is the "best place in town", i.e. the place which is closest to all consumers. This "minimum differentiation" arises because it is now the random choice factors that give the stores some market power, allowing them to raise the prices above cost prices. The larger the noise, the closer the stores locate to the centre of the "town". Eventually, at very large disorders, both stores locate at the center and are able to ask for high prices.

The original one dimensional Hotelling model has been extended to two dimensional (geographical) spaces by Veendorp and Majeed [6] in the absence of disorder. They reach a similar conclusion as other studies [7]: stores maximally differentiate in one of the dimensions, while adopting an identical location in the other dimension.

However, the case when the consumer choice is influenced by additional (and unobservable) factors, as mimicked by the logit function, has not been studied in two dimensions. This is the goal of the present paper. To the best of our knowledge, this paper presents the first analytical solution to Hotelling model in presence of noise in consumer's choice.

II. OPTIMAL STORE LOCATION IN A TWO DIMENSIONAL CITY

We consider a two-dimensional square city represented by $L \times L$ discrete sites occupied uniformly by consumers with unit density.

To model the behavior of the consumers, we assume that each consumer j can assign a "utility" function K_{js} to each store s given by:

$$K_{js} = R - p_s - a d_{js}^{\alpha} \tag{2.1}$$

where R is the "maximum utility" of buying the product, assumed to be high enough to prevent any negative value for K_{js}. p_s is the price of the product at store s, d_{js} the (Euclidian) distance between consumer j and store s, a is the transportation cost coefficient and α an exponent characterizing how transportation costs increase with distance. We take here $\alpha = 2$ for ease of calculations.

Consumers evaluate K_{js} for the two stores and choose according to these values. A simple way to include the effect of random factors that "smoothen" the consumer's choice among the stores is to use the well-known "logit" weighting factor [2]. We assign a probability for consumer j to buy at store s as:

$$\sigma_{js} = \frac{exp(K_{js}/T)}{\sum_{s'} exp(K_{js'}/T)} \tag{2.2}$$

T is a parameter which defines how sharply consumers discern between the expected surpluses offered by each site. As $T \to \infty$ consumers do not discriminate between eligible stores, whereas if $T \to 0$, consumers exclusively choose the store with the highest K_{js}.

Note that, the way the problem is posed, we have assumed that the probability that the consumers make a purchase is always 1 and the only decision concerns where the purchase is made.

In this situation, the average market M_s for store s is given by

$$M_s = \sum_j \sigma_{js} \tag{2.3}$$

and the expected profit accrued by this store will be given by

$$D_s = p_s M_s \tag{2.4}$$

III. ANALYTICAL SOLUTION

For definiteness, we list here the precise rules of the system under consideration:

- First, given any two positions of the stores, S_1 and S_2, the stores compete in prices until they reach (Nash) equilibrium; that is, until any unilateral change in price for either store leads to a lower profit. We further assume that the stores know the profits accrued by the equilibrium prices for every pair of store positions.
- Knowing the equilibrium prices for each pair of sites and the position of the other store, the stores then compete for optimal (maximal profit) location, until they reach (Nash) equilibrium in positions.

In our analysis we assume S_1 and S_2 to be continuous variables and, as mentioned above, we take $\alpha = 2$. Thus, M_1, the average market (number of consumers) attending store 1, is given by:

$$M_1 = \int_{-L/2}^{L/2} dj_x \int_{-L/2}^{L/2} dj_y \frac{1}{1 + e^{\{p_1 - p_2 + a[|J-S_1|^2 - |J-S_2|^2]\}/T}} \qquad M_2 = L^2 - M_1 \tag{3.1}$$

Given locations S_1 and S_2, price equilibrium is attained when

$$\frac{\partial D_1}{\partial p_1} = 0 \qquad \text{and} \qquad \frac{\partial D_2}{\partial p_2} = 0 \tag{3.2}$$

i.e. any unilateral change in the price offered by each store leads to a reduction of the profit gained by the store (of course, the above equations only reflect that at equilibrium profit is an extremum, this will do for the analysis because the extrema of these functions are maxima).

These are two coupled equations for the prices, from which we can determine, in principle, the prices as functions of the positions.

Now, equilibrium in the location competition will be achieved when neither store can increase its profit by changing its own position given the other store's location. Specifically, as mentioned above, storeowners know that a change in position will lead to a new set of prices, and they can evaluate the resulting change in profit accrued in the new position at the equilibrium prices corresponding to the new positions.

Thus, equilibrium is achieved when

$$\left(\frac{dD_1}{dx_1}, \frac{dD_1}{dy_1}\right) = 0 \qquad \text{and} \qquad \left(\frac{dD_2}{dx_2}, \frac{dD_2}{dy_2}\right) = 0 \tag{3.3}$$

where x_1, y_1 and x_2, y_2 are the components of S_1 and S_2 respectively. Regarding notation we choose to write the above equations as vectors of total derivatives with respect to the components of the positions of each store and reserve the gradient as a vector of partial derivatives to be used below.

From eq.(2.4), we can write the equation for price equilibrium for store 1 as:

$$p_1 \frac{\partial M_1}{\partial p_1} + M_1 = 0 \tag{3.4}$$

then, the condition for location equilibrium becomes

$$\left(\frac{dD_1}{dx_1}, \frac{dD_1}{dy_1}\right) = p_1 \frac{\partial M_1}{\partial p_2} \nabla_1 p_2 + p_1 \nabla_1 M_1 = 0 \tag{3.5}$$

where $\nabla_1 = (\partial/\partial x_1, \partial/\partial y_1)$ and we have used eq.(3.4). Note that, as required in the description of the location game, the above expression includes the change in p_2 due to the change of location of store 1. A corresponding equation holds for store 2:

$$\left(\frac{dD_2}{dx_2}, \frac{dD_2}{dy_2}\right) = p_2 \frac{\partial M_2}{\partial p_1} \nabla_2 p_1 + p_2 \nabla_2 M_2 = 0 \tag{3.6}$$

From the explicit expression of M_1, we have:

$$I \equiv \frac{\partial M_1}{\partial p_2} = \frac{1}{4T} \int_{-L/2}^{L/2} dj_x \int_{-L/2}^{L/2} dj_y \frac{1}{\cosh^2\left(\{p_1 - p_2 + a[|J - S_1|^2 - |J - S_2|^2]\}/2T\right)} \tag{3.7}$$

whereas

$$\nabla_1 M_1 = \frac{a}{2T} \int_{-L/2}^{L/2} dj_x \int_{-L/2}^{L/2} dj_y \frac{J - S_1}{\cosh^2\left(\{p_1 - p_2 + a[|J - S_1|^2 - |J - S_2|^2]\}/2T\right)} \tag{3.8}$$

Thus, the location equilibrium equation for store 1 can be written as

$$\nabla_1 p_2 + 2a[\langle J \rangle - S_1] = 0 \tag{3.9}$$

where $\langle J \rangle = (\langle j_x \rangle, \langle j_y \rangle)$ stands for

$$\langle J \rangle = \frac{\int_{-L/2}^{L/2} dj_x \int_{-L/2}^{L/2} dj_y J/\cosh^2\left(\{p_1 - p_2 + a[|J - S_1|^2 - |J - S_2|^2]\}/2T\right)}{\int_{-L/2}^{L/2} dj_x \int_{-L/2}^{L/2} dj_y 1/\cosh^2\left(\{p_1 - p_2 + a[|J - S_1|^2 - |J - S_2|^2]\}/2T\right)}$$

Now, $\nabla_1 p_2$ can be calculated from the conditions of price equilibrium (3.2). Writing them in terms of p_1, M_1, etc. (as in eq.(3.4)), and applying ∇_1 we have:

$$\begin{aligned}
&\left(2\frac{\partial M_1}{\partial p_1} + p_1 \frac{\partial^2 M_1}{\partial p_1^2}\right) \nabla_1 p_1 + \left(p_1 \frac{\partial^2 M_1}{\partial p_1 \partial p_2} + \frac{\partial M_1}{\partial p_2}\right) \nabla_1 p_2 + p_1 \nabla_1 \frac{\partial M_1}{\partial p_1} + \nabla_1 M_1 = 0 \\
&\left(2\frac{\partial M_2}{\partial p_2} + p_2 \frac{\partial^2 M_2}{\partial p_2^2}\right) \nabla_1 p_2 + \left(p_2 \frac{\partial^2 M_2}{\partial p_1 \partial p_2} + \frac{\partial M_2}{\partial p_1}\right) \nabla_1 p_1 + p_2 \nabla_1 \frac{\partial M_2}{\partial p_2} + \nabla_1 M_2 = 0
\end{aligned} \tag{3.10}$$

To simplify the above expressions, we use eq (3.5) and note the following identities:

$$\frac{\partial M_2}{\partial p_2} = \frac{\partial M_1}{\partial p_1} = -\frac{\partial M_2}{\partial p_1} = -\frac{\partial M_1}{\partial p_2} \equiv -I \qquad M_1 + M_2 = L^2 \tag{3.11}$$

where I is given in eq.(3.7).

Now we limit ourselves to search for equal price symmetric solutions i.e. solutions for which $S_1 = -S_2$; and, by symmetry, $p_1 = p_2$. Note that all the derivatives are to be taken *before* imposing the symmetry conditions. Under these conditons, the terms arising from the second derivatives with respect to the p's vanish (as well as $\langle j \rangle$)

Then, eqs.(3.10) become:

$$\begin{aligned} -2I\nabla_1 p_1 - p_1 \nabla_1 I = 0 \\ -2I\nabla_1 p_2 + I\nabla_1 p_1 - p_2\nabla_1 I - \nabla_1 M_1 = 0 \end{aligned} \tag{3.12}$$

From these equations we finally get

$$S = -\frac{3L^2}{4aI^2}[\nabla_1 I]_{s_2=-s_1} \tag{3.13}$$

Now, writing

$$S = L\sigma \qquad\qquad \gamma = 2aL^2/T \tag{3.14}$$

we can rewrite equation (3.13) explicitly in terms of the dimensionless vector σ and the parameter γ as

$$\sigma = 3\,\frac{\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} dj_x dj_y J \sinh[\gamma(J\cdot\sigma)]/\cosh^3[\gamma(J\cdot\sigma)]}{\left[\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} dj_x dj_y \; 1/\cosh^2[\gamma(J\cdot\sigma)]\right]^2} \tag{3.15}$$

These integrals can be performed explicitly:

$$\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} dj_x dj_y \; 1/\cosh^2[\gamma(J\cdot\sigma)] = \frac{2}{\gamma^2\sigma_x\sigma_y}\ln\left\{\frac{\cosh[\gamma(\sigma_x+\sigma_y)/2]}{\cosh[\gamma(\sigma_x-\sigma_y)/2]}\right\} \tag{3.16}$$

and

$$\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} dj_x dj_y J \sinh[\gamma(J\cdot\sigma)]/\cosh^3[\gamma(J\cdot\sigma)] = -\frac{1}{2\gamma}\nabla\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} dj_x dj_y \; 1/\cosh^2[\gamma(J\cdot\sigma)]$$

We further restrict our search to solutions along the diagonal: $\sigma_x = \sigma_y$, and along the horizontal: $\sigma_y = 0$.

For the horizontal case we obtain the following expression:

$$\sigma_x = \frac{3}{4}\left[\frac{\sinh(\gamma\sigma_x) - \gamma\sigma_x}{\cosh(\gamma\sigma_x) - 1}\right] \tag{3.17}$$

For $\gamma < 4$, the only solution for the above equation is $\sigma_x = 0$. For $\gamma > 4$, two other nontrivial solutions $\sigma_x = \pm\sigma_h^*(\gamma)$ appear. As $\gamma \to \infty$ (low T limit), it is easy to see that $\sigma_h^* \to 3/4$.

Along the diagonal, the equation for $\sigma_x = \sigma_y \equiv \sigma$ is

$$\sigma = \frac{6\gamma\sigma}{(4\ln[\cosh(\gamma\sigma)])^2}\left[2\ln[\cosh(\gamma\sigma)] - \gamma\sigma\tanh(\gamma\sigma)\right] \tag{3.18}$$

Once again, for $\gamma < 4$ the only solution to the above equation corresponds to $\sigma = 0$, whereas for $\gamma > 4$ we have two nontrivial solutions $\sigma = \pm\sigma_d^*(\gamma)$. Now, when $\gamma \to \infty$ we find $\sigma_d^* \to 3/8$.

Thus, for high enough T ($\gamma < 4$), the only solutions appear to be at the origin, i.e. at the center of the city. However, at lower T, at least 4 other solutions exist. These solutions move away from the center as $T \to 0$ as far as $3L/4$, which is beyond the boundary of the city (which is located at $L/2$) along the horizontal lines, but only as far as $3L/8$ along the diagonal lines.

To determine the optimal locations we need to evaluate the actual profits obtained by the stores for each of these solutions. This is relatively easy to do. First, we notice that for the equal price symmetric solutions we are considering, we have $M_1 = M_2 = L^2/2$. Thus, the equilibrium prices are given by $p_1 = p_2 = L^2/2I$ (where I is the integral defined in eq.(3.7) and which is trivially related to the integral in eq.(3.16)), and the profits are given by $D_1 = D_2 = L^4/4I$. Thus, we need to compare the behavior of I along the horizontal and diagonal solutions:

$$I(\sigma_h^*) = \frac{L^2}{2T\gamma\sigma_h^*} \tanh(frac\gamma\sigma_h^* 2) \equiv I_h(\gamma\sigma_h^*) \tag{3.19}$$

and

$$I(\sigma_d^*) = \frac{L^2}{2T\gamma^2{\sigma_d^*}^2} \ln[\cosh(\gamma\sigma_d^*)] \equiv I_d(\gamma\sigma_d^*) \tag{3.20}$$

When solving eqs. (3.17) and (3.18) and using these to evaluate the profits, it appears that for $\gamma > 4$ the profit found with the horizontal solution is always higher. For $\gamma \leq 4$ both solutions are at the origin and profits are, of course, equivalent. These results are plotted in Figures 1 and 2.

IV. SIMULATION RESULTS

To confirm the previous treatment, we performed numerical simulations under the same hypothesis. Specifically, for each value of γ, we performed a double nested optimization, using brent's minimization technique [9]. The town size (L) is taken as 300.

One loop calculates the profits given the stores locations σ_1 and σ_2.

- Given σ_1 and σ_2, and an initial value for p_2, calculate p_1^* so as to maximize D_1 using brent's algorithm
- Given σ_1 and σ_2, and p_1^*, calculate p_2^* so as to maximize D_2 using brent's algorithm
- Repeat the two preceding steps until p_1^* and p_2^* converge within a fixed tolerance

Numerically, the relative tolerance value for the prize was taken as 10^{-6}.

To find the optimal stores positions, a similar loop is then used, which includes the first loop at each step, to calculate the profits.

- Given an initial value for σ_2, calculate σ_1^* so as to maximize D_1 using brent's algorithm
- Given σ_1^*, calculate σ_2^* so as to maximize D_2 using brent's algorithm
- Repeat the two preceding steps until σ_1^* and σ_2^* converge within a fixed tolerance

Numerically, the relative tolerance value for the position was taken as 10^{-5}.

Optimal locations and corresponding profits as a function of γ are plotted in Figures 1 and 2. The agreement with the theoretical predictions is almost perfect, except in the region $\gamma = 4$ where discreteness effects seem to play a role. This is confirmed by the results obtained for a larger number of customers (600 x 600 or 1000 x 1000), which are closer to the theoretical results (see Fig. 3).

V. DISCUSSION

In this model, equilibrium configurations result from trying to maximize individual profits, which are the product of price and market. This last quantity, the market of a store, is a decreasing function of the price offered at that store and of distance to the consumers, and a growing function of the price offered by the competition. This combination can give rise to rather unexpected strategies: For example, at low temperatures, when consumers buy almost exclusively at the store which maximizes their utility, a store can find it profitable to reduce its market by increasing its price and moving away from the city center because the other store will now find it profitable to increase its own price, thus returning part of the market to the first store. At this point, the second store might find it even more profitable to move further away from the center and ask for an even higher price, knowing that the first store will also increase its price. This process can go on until the transportation cost becomes too large to be compensated by the other store's price increase, at which point, the off center equilibrium positions are attained.

On the other hand, as the temperature rises, consumers become more indiferent to the travelling cost and stores can gain more by undercutting the competition as long as no consumers are forced to travel too long a distance to make their purchase. This brings the stores closer to the center of the city even though the price competition brings the profits down.

Eventually, the cost of transportation becomes almost irrelevant, and purchases are mostly decided by the price offered at each store. At the higher values of T, even price is relatively unimportant and there is little to win by undercutting the competition so prices and profits grow again.

Another aspect that has drawn a great deal of attention is the fact that the spatial differentiation occurs in one dimension and not in the other [7]. Clearly this a result of the chosen shape of the city and on a circular city no such thing would happen. Now, admittedly, we did not search analytically for equilibrium positions out of the axes of symmetry, yet we did not find any in our numerical simulations. Thus, given the geometry we considered, the question that remains is why are the equilibria along the paralel axis more profitable than those along the diagonals. A partial explanation for this might be that if one projects the market along the diagonal, it becomes apparent that most of the market is at the middle and diminishes towards the corners. Thus, the loss of market incurred by the first store moving away from the center affects more consumers. Further, to render such a movement profitable, the store must also increase its price. However, this increase must be modest if the combination of movement and price increase is to be compensated by the second store's price hike.

VI. CONCLUSION

In this article we extend the Hotelling model by studying the optimal location of two stores in a two-dimensional city where consumers' choice follows a logit function with parameter T. Starting from an identical position in the center of town when T is large, stores differentiate only along one coordinate as T decreases. Both prices and profits increase as T increases, i.e. as the consumer pay less and less attention to the difference in costs incurred by attending one or the other store. The results are therefore similar to the one dimensional case.

An interesting but much more complicated extension of the model would be to consider the situation in which the consumers, with certain probability, may choose not to make the purchase. For example, one could assume that it is highly improbable that a consumer makes a purchase if the cost incurred in attending either store is too high. Among other things, such an extension would introduce a T dependent monopoly price to the single store system, as well as another distance defining the market ranges of each store; i.e. finite regions around each store beyond which it is high unlikely that consumers will attend the store. This line of research is currently in progress.

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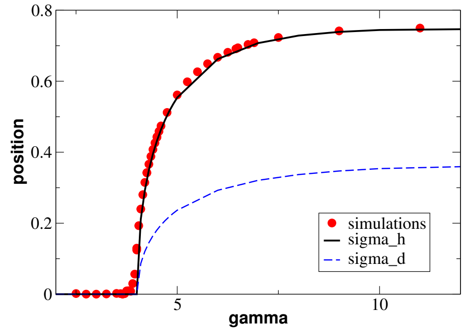

FIG. 1. Equilibrium positions as a function of γ.

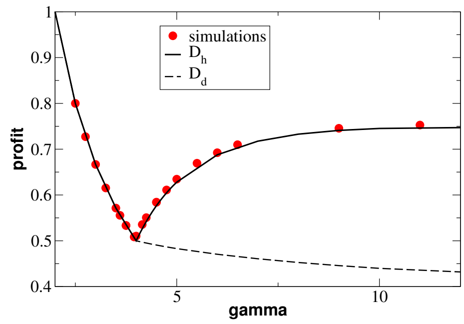

FIG. 2. Store profits at the equilibrium positions as a function of γ

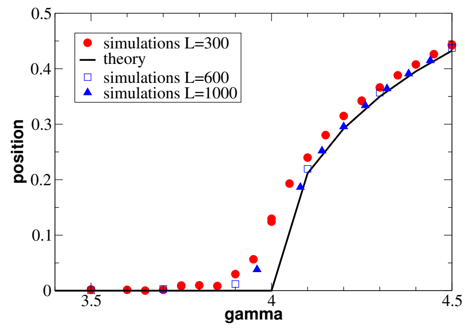

FIG. 3. Detailed view of store positions close to the transition occuring at $\gamma = 4$